Quantum effects for the 2D soliton in isotropic ferromagnets

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We evaluate a zero–point quantum correction to a Belavin–Polyakov soliton in an isotropic 2D ferromagnet. By revising the scattering problem of quasi–particles by a soliton we show that it leads to the Aharonov–Bohm type of scattering, hence the scattering data can not be obtained by the Born approximation. We proof that the soliton energy with account of quantum corrections does not have a minimum as a function of its radius, which is usually interpreted as a soliton instability. On the other hand, we show that long lifetime solitons can exist in ferromagnets due to an additional integral of motion, which is absent for the σ–model.

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Solitons are known to play an important role in several branches of field theory and condensed matter physics, see Ref. 1 for a review. In particular, solitons treated as nonlinear excitations are important in 1D and 2D magnetism [2, 3, 4]. A serious impediment in studying 2D spin systems arises due to the absence of exact analytical solutions for most models. Thereupon special attention is deserved to models which admit an analytical treatment. One of the well–known examples is a model of the 2D isotropic ferromagnet (FM), which provides an exact analytical soliton, the so–called Belavin–Polyakov (BP) soliton [5]. In terms of the normalized magnetization, $\boldsymbol{m} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, the soliton structure $\boldsymbol{m}_{\text{BP}}$ is described by the formula [5]

$$\tan\frac{\theta_{\text{BP}}}{2} = \left(\frac{R}{\rho}\right)^{|q|}, \qquad \phi_{\text{BP}} = \varphi_0 + q\chi. \tag{1}$$

Here ρ and χ are the polar coordinates in the magnet plane, the integer q is the π_2–topological charge of the soliton, R and φ_0 are arbitrary parameters. BP–solitons of the form (1) appear in different models of non-linear field theory and condensed matter physics [1]. In particular, the BP–solitons are important for the ferromagnetic quantum Hall effect [6].

The unique problem of the BP soliton is that its energy $\mathscr{E}_{\text{BP}} = 4\pi JS^2|q|$ (J is the exchange integral, S is the atomic spin) does not depend on its radius R: this results from the scale invariance of the system, which is a part of the general conformal invariance of the model. Since the soliton radius is not fixed, one is then free to let it go up to the system size, and the thermal excitation of solitons will break the long-range order [5]. However, recent studies have shown that a quantization of the soliton of the classical σ-model, which can be attributed to antiferromagnets, breaks the static scale invariance [7, 8, 9]. The natural question whether it works for FMs is still open.

The purpose of our study is to examine the role of quantum fluctuations for the soliton properties. We treat the problem semiclassically using the one–loop correction to the classical soliton energy, originally calculated by Dashen et al. [10, 11] for 1D solitons, see also Ref. 12. To generalize these results to the 2D case one needs to solve the soliton–magnon scattering problem for 2D magnets. We show a *unique new feature* of the 2D soliton–magnon interaction, which is absent in 1D: the soliton acts on magnons not only by some local potential, but also in the same way as an effective long–ranged magnetic field acts on a charged particle; this essentially changes the scattering picture, leading to the Aharonov–Bohm (AB) scenario. We state that the AB type of scattering is a general consequence of 2D scattering by a topological soliton. We calculate the Casimir energy for the BP–soliton in FMs and show that the quantum correction can not provide a fixed size for the soliton. Nevertheless, we show that long lifetime solitons can exist in FMs due to an additional integral of motion, contrary to antiferromagnets.

The macroscopic dynamics of the classical FM follows the Landau–Lifshitz equations

$$\begin{aligned} \frac{1}{D}\sin\theta\,\partial_t\phi &= \boldsymbol{\nabla}^2\theta - \sin\theta\cos\theta\,(\boldsymbol{\nabla}\phi)^2\,, \\ \frac{1}{D}\sin\theta\,\partial_t\theta &= -\boldsymbol{\nabla}\cdot\left(\sin^2\theta\boldsymbol{\nabla}\phi\right), \end{aligned} \tag{2}$$

where D is the stiffness coefficient of the spin–waves, which are characterized by the dispersion law $\omega(\boldsymbol{k}) = Dk^2$. To analyze the soliton–magnon interaction, we consider small oscillations of the magnetization $\boldsymbol{m}$ on the background of the stationary BP–soliton $\boldsymbol{m}_{\text{BP}}$. These oscillations can be described in terms of the complex valued "wave function" $\Psi = \theta - \theta_{\text{BP}} + i\sin\theta_{\text{BP}}(\phi - \phi_{\text{BP}})$, see Ref. 13. For the further analysis it is instructive to rewrite the linearized equation for the Ψ–function in the

form of the Schrödinger equation:

$$H\Psi = \frac{i}{D}\partial_t \Psi, \quad H = (-i\boldsymbol{\nabla} - \boldsymbol{A})^2 + V, \tag{3a}$$

$$V = -\frac{q^2}{\rho^2}\sin^2\theta_{\mathrm{BP}}, \qquad \boldsymbol{A} = -\frac{q\cos\theta_{\mathrm{BP}}}{\rho}\boldsymbol{e}_\chi. \tag{3b}$$

The Hamiltonian H has a form which is typical for a quantum–mechanical charged particle in the presence of a scalar potential V and an additional magnetic field with a vector potential $\boldsymbol{A}$. The appearance of an effective magnetic field is a new feature of the 2D soliton–magnon interaction, which is always absent in 1D systems. Discerning this effective magnetic field gives the possibility to draw a number of general conclusions about soliton–magnon scattering in the 2D case, see below.

For the system (3) we apply the standard partial wave expansion, using the *Ansatz*:

$$\Psi(\rho,\chi,t) = \sum_{\alpha=(k,m)} \psi_m(\rho)\exp\left(im\chi - i\omega_\alpha t + \beta_\alpha\right). \tag{4}$$

Here the integer m is the azimuthal quantum number, k is the radial wave number, and β_α is an arbitrary initial phase. Each partial wave ψ_m is an eigenfunction of the 2D radial Schrödinger equation

$$\begin{aligned}&\left(-\nabla_\rho^2 + U_m\right)\psi_m = k^2\psi_m,\\ &U_m = \frac{m^2 + 2mq\cos\theta_{\mathrm{BP}} + q^2\cos 2\theta_{\mathrm{BP}}}{\rho^2}.\end{aligned} \tag{5}$$

Here the term linear in m reminds of an effective magnetic field $\boldsymbol{A}$. The scattering problem can be formulated in the usual way. The eigenfunctions for free magnon modes have the form $\psi_m^{\mathrm{free}} \propto J_{|m|}(k\rho)$, with an asymptotic behavior $\psi_m^{\mathrm{free}} \propto (k\rho)^{-1/2}\cos(k\rho - |m|\pi/2 - \pi/4)$ when $k\rho \gg |m|$; J_m is Bessel function. In the presence of a soliton the behavior of a magnon solution can be analyzed at large distances, $\rho \gg R$. In view of the asymptotic behavior $U_m \approx |m+q|^2/\rho^2$, in the limiting case $k\rho \gg |m|$ one has the usual result [13]:

$$\psi_m \propto \frac{1}{\sqrt{k\rho}}\cos\left(k\rho - \frac{|m+q|\pi}{2} - \frac{\pi}{4} + \eta_m(k)\right).$$

The phase shift η_m contains all information about the scattering process.

The main features of the scattering on a topological BP–soliton are caused by the magnetic field $\boldsymbol{A}$. As an analogue of the Zeeman splitting of electron energy terms in an external magnetic field, the presence of an effective magnetic field breaks the symmetry $\eta_m(k) = \eta_{-m}(k)$. It is necessary to take into account separately positive and negative m's.

Sometimes the soliton–magnon scattering problem is treated perturbatively using the Born approximation [7, 9], which, in principle, can be used for the scattering in a magnetic field. However, due to the topological soliton properties, $\boldsymbol{A}$ is a long–ranged field

$$\boldsymbol{A}(\rho) = \frac{1-\left(\dfrac{\rho}{R}\right)^{2|q|}}{1+\left(\dfrac{\rho}{R}\right)^{2|q|}}\frac{q}{\rho}\boldsymbol{e}_\chi \sim \begin{cases} +\dfrac{q}{\rho}\boldsymbol{e}_\chi & \text{when } \rho \ll R,\\ -\dfrac{q}{\rho}\boldsymbol{e}_\chi & \text{when } \rho \gg R,\end{cases} \tag{6}$$

which is typical for the AB effect [14]. For such a type of scattering, some standard scattering results fail. For example, the Levinson theorem must be modified for long–range potential systems [14, 15]. Since scattering phase shifts are not still localized, there appears a problem of the regularization of the scattering series like in conventional AB scattering picture [16]. As was firstly noted by Feinberg [17], the Born approximation fails for the AB scattering; it gives an average of two different modes with opposite signs of m [18]. Thus we need a more precise approach than the Born approximation. One needs to stress that such a long–range behavior is not a result of slow algebraic decay of the out–of–plane structure of the BP soliton. It is a consequence of the topology of the BP–soliton, namely, of the relation $\phi = q\chi$, thus the AB–type of scattering is valid also for anisotropic magnets [19, 20, 21].

Let us discuss the soliton with the topological charge $q = 1$, which has the lowest energy. Such a soliton has two internal zero–frequency modes, which are the limit of the continuum spectrum as $k \to 0$: [22]

$$\psi_{m=+1}^{(k=0)} = \frac{1}{\rho^2 + R^2}, \qquad \psi_{m=0}^{(k=0)} = \frac{\rho}{\rho^2+R^2}. \tag{7}$$

The mode with $m = +1$ is a local translational mode, which describes a soliton shift, the mode with $m = 0$ is the half–local rotational mode. The mode with $m = +1$ has an exact analytical solution for any finite values of k

$$\psi_{m=1}(\rho) = J_2(k\rho) - \frac{2}{k\rho}\frac{J_1(k\rho)}{(\rho/R)^2+1}, \tag{8}$$

hence this mode does not scatter at all, $\eta_{m=+1} = 0$ [13]. Note that in the interesting case of long–wavelength asymptotic behavior ($k\rho \ll 1$) at large distances $\rho \gg R$, this expression has the same form as a combination of Bessel and Neumann functions, $J_2(k\rho) \propto (k\rho)^2$ and $Y_2(k\rho) \propto (k\rho)^{-2}$. Thus, the second term in (8) imitates the presence of the function Y_2 and the presence of scattering, which caused a conclusion in Ref. 8 that the mode with $m = +1$ can be scattered. The scattering phase shift can be found in both limiting cases, for small and large dimensionless radial wave number $\varkappa = kR$ [13]. For long–wave lengths, $\varkappa \ll 1$

$$\eta_m \underset{\varkappa\ll 1}{\sim} \begin{cases} 0, & \text{when } m=1,\\ -\dfrac{\pi}{2\ln(1/\varkappa)}, & \text{when } m=0,\\ -\pi\varkappa^2\ln\dfrac{1}{\varkappa}, & \text{when } m=-1,\\ \dfrac{\pi\varkappa^2}{2m(m+1)}\operatorname{sgn} m, & \text{otherwise,}\end{cases} \tag{9a}$$

and in the opposite case of short–wave lengths [13]

$$\eta_m \underset{\varkappa\gg 1}{\sim} \pi \,\mathrm{sgn}\,(m-1) - \frac{\pi(m-1)}{\varkappa}. \tag{9b}$$

Now we are able to calculate the density of magnon states. Let us generalize the main arguments of Dashen et al. [10, 11] for the 2D system. The idea of the approach is to calculate energy shifts of vacuum magnon states in the presence of a soliton, which are constructed as one-loop quantum corrections to the soliton energy. The energy of the vacuum comes from the zero-point fluctuations of the magnon states. Without the soliton, each vacuum magnon makes a contribution as $\hbar D\boldsymbol{k}_{\mathrm{vac}}^2/2$, where $\{\boldsymbol{k}_{\mathrm{vac}}\}$ is set of allowable wave vectors. In the soliton presence the set of allowable wave vectors changes, $\{\boldsymbol{k}\}$. The energy of the state with the wave vector $\boldsymbol{k}$ is $\hbar D\boldsymbol{k}^2/2$. Therefore the energy correction is

$$\mathscr{E}^{1\text{–loop}} = \frac{\hbar D}{2}\sum_{\boldsymbol{k}} \boldsymbol{k}^2 - \frac{\hbar D}{2}\sum_{\boldsymbol{k}^{\mathrm{vac}}} \boldsymbol{k}_{\mathrm{vac}}^2. \tag{10}$$

To determine the set of allowable states, we put the system in a very large box of the size L, making all states discrete. In the limiting case $L\to\infty$ the energy correction (10) does not depend on the form of the boundary conditions. For the 2D case we choose fixed boundary conditions for a circular box of radius L [13]. Since the free magnons are described by the Bessel function ψ_m^{free}, by enforcing fixed boundary conditions $\psi_m^{\mathrm{free}}(\rho = L) = 0$, we fix the allowed values of the radial wave number, $k_n^{\mathrm{vac}}L = j_m^{(n)}$, where $j_m^{(n)}$ is the n–th zero of the Bessel function J_m. In the region of interest, $n\gg 1$, the zeros of the Bessel function $j_m^{(n)} \approx \pi n$. Thus the allowed values of the wave numbers are $k_n^{\mathrm{vac}} \approx \pi n/L$, similar to the 1D case. However, the above–used simple equation for $j_m^{(n)}$ is valid only if $|m|$ is not very large. In the case of $|m|\gg 1$, the first zero of the Bessel function $j_m^{(1)} \approx |m|$. Hence, in a finite system there appears a restriction for the allowed number of modes, $|m|\leq L$, and the sum rule for the 2D case takes the form:

$$\sum_{k,m}\Bigl(\bullet\Bigr) \longrightarrow \frac{L}{\pi}\int_0^\infty \mathrm{d}k \sum_{m=-kL}^{kL}\Bigl(\bullet\Bigr). \tag{11}$$

For magnon states in the presence of the soliton there appears a phase shift η_m due to the soliton–magnon scattering, therefore

$$k_nL + \eta_m = k_n^{\mathrm{vac}}L = j_m^{(n)} \qquad \Longrightarrow \qquad k_n - k_n^{\mathrm{vac}} = -\frac{\eta}{L}.$$

The one-loop correction to the soliton energy reads

$$\mathscr{E}^{1\text{–loop}} = -\frac{\hbar D}{\pi}\int_0^\infty k\,\eta(k)\mathrm{d}k, \quad \eta(k) = \sum_{m=-kL}^{kL}\eta_m(k). \tag{12}$$

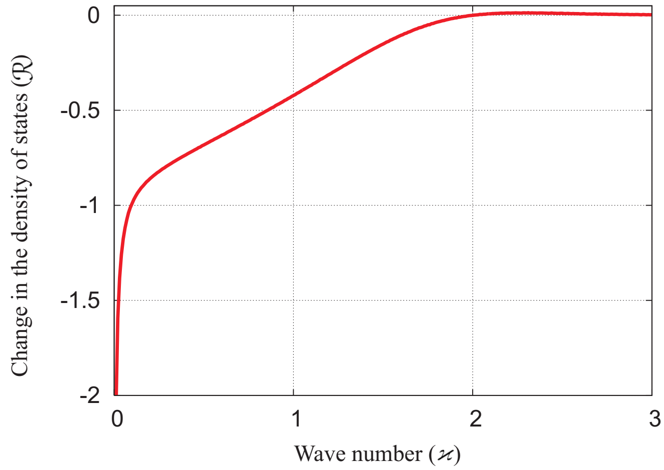

FIG. 1: The change $\mathcal{R}$ in the density of magnon states due to the soliton *vs* dimensionless wavenumber ($\varkappa = kR$), obtained by numerical integration of the Schrödinger equation (5).

Here $\eta(k)$ is a sum of the phase shifts of all partial waves with the fixed radial wave number k. Note that our definition of $\eta(k)$, in contrast to the case of the σ-model [9], contains independent summations over positive and negative m, which reflects the breaking of the symmetry $m\to -m$.

The short–wave length behavior of the phase shift is responsible for ultraviolet (UV) singularities. Following Refs. 9 in order to avoid the UV singularity one needs to derive the short–wave length asymptotics for the phase shift. There appears a problem to sum to infinity an alternating series, which has no absolute convergence. Symmetric limits regularize this summation similar to the exponential regularization for the original AB–scattering [16]; different waves are taken into account in the order in which the poles $k_m^{(p)}$ in the scattering amplitude appear as k increases [24]. Finally, we found that $\eta(\infty) = -\pi$ for the soliton with $q = 1$[25]. Note that this result can not be obtained from the Born approximation because of the long–range nature of the AB–scattering; namely, perturbative Born calculations resulted in different conclusions that $\eta(\infty) = \pi$ (see Ref. 7) and $\eta(\infty) = 2\pi$ (see Ref. 9).

To check our analytical predictions we have calculated $\eta(k)$ by the numerical integration of the Schrödinger equation (5) for $m\in[-100;100]$, which gives $\eta(\infty) = -\pi$ with the precision 10^{-3}.

Finally, the one-loop correction reads $\mathscr{E}^{1\text{–loop}} = \mathscr{E}_{\mathrm{CT}} + \mathscr{E}^{\mathrm{Cas}}$. Here the counterterm $\mathscr{E}_{\mathrm{CT}} = -\frac{\hbar D}{2\pi}\eta(k)k^2\big|_0^\infty$ can be evaluated using a momentum cutoff Λ, which results in $\mathscr{E}_{\mathrm{CT}} = \frac{1}{2}\hbar D\Lambda^2$. Taking away this UV term, e.g. by renormalizing the exchange constant [9], we ends with

the finite Casimir energy:

$$\mathscr{E}^{\mathrm{Cas}} = \frac{\hbar D}{2R^2}\int\limits_0^\infty \varkappa^2 \mathcal{R}(\varkappa)\mathrm{d}\varkappa,\ \mathcal{R}(\varkappa) = \frac{1}{\pi}\sum_{m=-\varkappa L/R}^{\varkappa L/R}\frac{\mathrm{d}\eta_m(\varkappa)}{\mathrm{d}\varkappa}. \tag{13}$$

Here $\mathcal{R}(\varkappa)$ describes the change in the density of magnon states due to the soliton. This expression can be analyzed analytically in limiting cases. Using the asymptotical behavior for the phase shift for different modes one can conclude that the maximum scattering in the long wavelength limit corresponds to the mode with $m = 0$, hence the density of states has a singularity: $\mathcal{R}(\varkappa) \sim -(2\varkappa)^{-1}\ln^2\varkappa \to -\infty$ when $\varkappa \to 0$. In the short wavelength limit all modes compensate each other and $\mathcal{R}(\varkappa \to \infty) \to 0$. In the intermediate range the change in the density of states can be found numerically only, see Fig. 1. One can conclude that for all $\varkappa$ the change in the density of states takes only negative values and $\int_0^\infty \mathcal{R}(\varkappa)\mathrm{d}\varkappa = -1$. Finally, the Casimir energy is obtained as

$$\frac{\mathscr{E}^{\mathrm{Cas}}}{\mathscr{E}^{\mathrm{BP}}} = -\frac{C}{8\pi S}\left(\frac{a}{R}\right)^2, \quad C = \int\limits_0^\infty \varkappa^2\left|\mathcal{R}(\varkappa)\right|\mathrm{d}\varkappa, \tag{14}$$

where C is a constant, which we calculated numerically to be $C \approx 0.38$. We introduced in (14) the typical length scale $a = \sqrt{\hbar D/JS}$, which is about a lattice constant. Note that this parameter is absent for the static BP–soliton problem, but it naturally breaks an initial scale invariance of the model in the dynamics, see Eq. (2). The soliton energy is reduced when the soliton radius decreases.

Let us discuss physical consequences of Eq. (14). First, the soliton energy with account of the quantum correction does not have a minimum as a function of the soliton radius. Usually this is interpreted as a soliton instability in the context of the Hobart–Derrick theorem, see Ref. 1. We will show here that this property leads to a dissipation of the soliton energy caused by magnon radiation, common to that for 3D Hopf solitons in isotropic FMs [23]. As an important contrast to σ–models, a model of the FM has an additional integral of motion, the z–component of the total spin $S_z \approx 2\pi S(R/a)^2$ [2]. Even for the static limit S_z takes a nonzero value; the energy dissipation caused by the radiation of magnon pairs with wave vectors $\boldsymbol{k}$ and $-\boldsymbol{k}$ will be accompanied by a decrease of the value of S_z by two. Therefore the soliton lifetime $\tau = S_z/(\mathrm{d}S_z/\mathrm{d}t)$ can be sizeable, when S_z is large (S_z is the number of bound magnons in the soliton).

The amplitude of the radiation process is $\varpi \sim JS(ak)^2$, see [23]. In accordance to Fermi's golden rule,

$$\frac{\mathrm{d}S_z}{\mathrm{d}t} = \sum_k 2\pi\frac{|\varpi|^2}{\hbar}\delta\left(\frac{\mathrm{d}E}{\mathrm{d}S_z} - \hbar\omega(k)\right).$$

Here E is the total energy of the soliton with account of the Casimir energy. Calculating $\mathrm{d}S_z/\mathrm{d}t$ in the continuum limit, one can finally obtain the soliton lifetime

$$\tau \sim \frac{4\pi\hbar}{JC^2}\left(\frac{R}{a}\right)^{10}. \tag{15}$$

Note that the lifetime is much bigger than $\hbar/J$ for $R > a$.

To conclude, quantum effects decrease the energy of the BP soliton in isotropic FMs, more strongly for small soliton radius. Nevertheless, the original argumentation by Belavin and Polyakov [5] about the breaking of the long-range order of the system is still valid. It is based on the fact that in isotropic FMs the energy is independent of the soliton radius R, and R can be comparable with the system size, $R \sim L$. However, in the case of large radii the quantum correction is negligible, the energy has a finite limit for $R \to \infty$, so the problem of long-range ordering has the classical form. Another aspect of the problem is the fate of the BP soliton at small finite R. As can be seen form Eq. (15), the lifetime is small for small R. In some respects the situation is similar to the problem of the black hole evaporation, i.e. the large radius soliton dissipation is very slow, and can be neglected. At the same time the dissipation of small radius solitons is very fast. When the soliton radius is small enough, the speed of dissipation increases rapidly; in the final stage with $S_z \sim S$ the soliton can collapse by a quantum jump, which is accompanied by a change of the topological charge.

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[24] The pole of the scattering amplitude corresponds to the condition when $\eta(k_m^{(p)}) = \pm\pi/2$. Asymptotically $k_m^{(p)} \sim |m|/R$ when $k >> 1/R$, see details in Ref. 13.

[25] More general, $\eta(\infty) = -\pi|q|$ for higher topological charges [14].